Monoidal structure of the category of u_q^+-modules

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1 Introduction.

We consider the half-quantum group $\mathrm{u}_q^+(\mathfrak{sl}_2(\mathbb{C}))$ at a root of unity which order is not 4. This non quasi-cocommutative Hopf algebra is the upper triangular sub-Hopf algebra of $\mathrm{u}_q(\mathfrak{sl}_2(\mathbb{C}))$, quotient of the quantized enveloping algebra at a root of unity q (see [5]). Half quantum groups provide universal R-matrices through the Drinfeld double and hence solutions to the Yang-Baxter equation. Furthermore they appear of interest in knot theory and 3-manifold invariants. For a simple Lie algebra $\mathfrak{G}$, a presentation of $\mathrm{u}_q^+(\mathfrak{G})$ by quiver and relations has been established by Cibils in [3], showing that only u_q^+ is of finite representation type, the others being of tame or wild representation type.

In order to study more deeply the representation theory of u_q^+, we consider the particular family of indecomposable modules on u_q^+ which are u_q-modules as well. We call them "extendable modules" . They form a subring of the Grothendieck ring of u_q^+, and their study leads to a Clebsch-Gordan-like formula for the decomposition of the tensor product, taken on the ground field, of two indecomposable u_q^+-modules. The extendable modules, together with the R-matrix of u_q and the action of the Auslander-Reiten transpose (see [1]) on the category of modules, complete the proof which was not achieved in [2]. As a consequence the tensor product commutes, despite the non quasi cocommutativity of u_q^+. Moreover we obtain explicit isomorphisms between $M \otimes N$ and $N \otimes M$ for any two u_q^+-modules and we can observe that these canonical isomorphisms have the properties of morphisms in a braided category (see [5]), except of course that they are not natural.
We also consider tensor products of simple modules over the entire u_q. The crucial observation is that extendable non-projective u_q^+-modules are the simple modules on u_q. A connection between the decomposition formulas over u_q^+ and u_q is established. We thus derive formulas previously obtained

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by Reshetikhin and Turaev in [8] for the tensor product of simple u_q-modules in a new way. The proof we obtain is new and entirely based on basic properties of extendable modules.

Furthermore we establish a totally different proof of the decomposition formula for u_q^+-modules which actually includes the three situations u_q^+, the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ and the quantum universal enveloping algebra $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ when q is not a root of unity. The proof consists in a fairly simple axiomatisation on the Grothendieck ring of these Hopf algebras.

2 The Hopf algebras $\mathrm{u}_q(\mathfrak{sl}_2(\mathbb{C}))$ and u_q^+.

We recall definitions and known facts about the above algebras, choosing Kassel's (see [5]) presentation of u_q, originally from Lusztig (see [7]). Let q be a primitive n-th root of unity in $\mathbb{C}$, n different from 4, and set

$$d = \begin{cases} \mathrm{n} & \text{if n is odd} \\ \mathrm{n}/2 & \text{if n is even} \end{cases}$$

Definition 2.1 *The Hopf algebra $\mathrm{u}_q(\mathfrak{sl}_2(\mathbb{C}))$ is defined over $\mathbb{C}$ by the generators E, F, K and the relations :*

$$E^d = F^d = 0, \quad K^d = 1, \quad KE = q^2 EK, \quad KF = q^{-2} FK$$

$$\text{and} \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

It admits a Poincar-Birkhoff-Witt type basis in the set $\{E^i K^j F^l\}$ for $0 \leq i, j, l \leq d-1$ (see [5]).
The coalgebra structure is given on the generators as follows :
the comultiplication $\Delta : \mathrm{u}_q \longrightarrow \mathrm{u}_q \otimes \mathrm{u}_q$ is defined by

$$\begin{aligned} \Delta(E) &= 1 \otimes E + E \otimes K \\ \Delta(F) &= K^{-1} \otimes F + F \otimes 1 \\ \Delta(K) &= K \otimes K, \end{aligned}$$

the counit $\epsilon : \mathrm{u}_q \longrightarrow k$ by

$$\epsilon(E) = \epsilon(F) = 0 \quad \epsilon(K) = 1$$

and the antipode, $S : \mathrm{u}_q \longrightarrow \mathrm{u}_q$, is given by

$$S(E) = -EK^{-1}, \; S(F) = -KF, \; S(K) = K^{-1}$$

We have the following formulas for the comultiplication :

$$\Delta(E)^r = \sum_{k=0}^{j} q^{-k(r-k)} \begin{bmatrix} r-k \\ r \end{bmatrix}_q E^k \otimes K^k E^{r-k} \quad \text{and}$$

$$\Delta(F)^r = \sum_{k=0}^{r} q^{k(r-k)} \begin{bmatrix} r-k \\ r \end{bmatrix}_q F^k K^{-(r-k)} \otimes F^{r-k}$$

Where $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{[y]!}{[x]![y-x]!}$ with $[x]! = [1][2]\dots[x]$ and $[x] = \frac{q^x - q^{-x}}{q-q^{-1}}$. A formula which calculates the commutators $[E^m, F^m]$ with $m \in \{0, \dots, d-1\}$ will be needed (see [5]):

$$E^m F^m = \sum_{h=0}^{m} c_h F^{m-h} E^{m-h} \prod_{j=0}^{h-1} \frac{Kq^{-j} - K^{-1}q^j}{q - q^{-1}}$$

where c_h is a nonzero coefficient. It is well known that this Hopf algebra is quasi-triangular (see [5], [6]). Its R-matrix has the following expression :

$$R = \frac{1}{d} \sum_{0 \leq i,j,k \leq d-1} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E^k K^i \otimes F^k K^j$$

Remark 2.1 1) Hopf algebras have the property that the tensor product over the ground field of two left modules is still a left module. Indeed, for a Hopf algebra H, restricting the natural action of $H \otimes H$ to H through the comultiplication Δ yields a left H-module structure.
2) Recall that the R-matrix satisfies in particular the relation $\Delta^{op} = R\Delta R^{-1}$, where $\Delta^{op} = \tau\Delta$ and τ is the flip, $\tau(a \otimes b) = b \otimes a$ for $a, b \in \mathrm{u}_q$. This relation is equivalent to the existence of a family of natural isomorphisms between $M \otimes N$ and $N \otimes M$ for any u_q-modules M and N. The isomorphisms are given by the action of τR.

The upper triangular sub-algebra of u_q generated by E and K is a sub-Hopf algebra, denoted by u_q^+ ; indeed
$\Delta(\mathrm{u}_q^+) \subset \mathrm{u}_q^+ \otimes \mathrm{u}_q^+$
$S(\mathrm{u}_q^+) \subset \mathrm{u}_q^+$.
The dimension over k of u_q^+ is d^2. The set $\{E^i K^j\}_{0 \leq i,j \leq d-1}$ is a basis of u_q^+ (see [7]).

Remark 2.2 In [2] it has been shown that u_q^+ is isomorphic to a quotient of a path algebra endowed with a Hopf algebra structure. It is our reference for the following remarks as well as for the representations of u_q^+.
1) As an associative algebra u_q^+ is uniserial, meaning that each indecomposable module on u_q^+ has a unique decomposition series. As a consequence u_q^+ is of finite representation type.
2) The Jacobson radical of u_q^+ is generated by E.

We have the following proposition :

Proposition 2.1 *If q is an n-th root of unity with $n \neq 4$, the Hopf algebra u_q^+ is not quasi-cocommutative.*

Proof : Suppose there exists an invertible element $R \in \mathrm{u}_q^+ \otimes \mathrm{u}_q^+$ such that $\Delta^{op} = R\Delta R^{-1}$. Then R is of the form $R = \sum_{0\le i,j,k,l\le d-1} a_{i,j,k,l} E^i K^j \otimes E^k K^l$ where the $a_{i,j,k,l}$ belong to k. We have in particular $\Delta^{op}(K) = R\Delta(K)R^{-1}$, i.e. $K \otimes KR = RK \otimes K$, implying that $a_{i,j,k,l} q^{2(i+k)} = a_{i,j,k,l}$. Hence the expression of R must reduce to
$R = \sum_{0\le i,j,l\le d-1} a_{i,j,d-i,l} E^i K^j \otimes E^{n-i} K^l \ + \ \sum_{0\le j,l\le n-1} a_{0,j,0,l} K^j \otimes K^l$.
In order to show that the coefficients $a_{o,j,o,l} = a_{j,l}$ are 0 we use the identity $\Delta^{op}(E)R = R\Delta(E)$ and obtain the relations
$a_{j,l} = q^{2j} a_{j,l-1}$ and $a_{j,l} = q^{-2l} a_{j-1,l}$ whenever they make sense. As a consequence, $a_{j,o} = a_{0,l} = a_{0,0}$, implying that $a_{1,1} = q^2 a_{1,0} = q^{-2} a_{0,1}$, hence $a_{1,1} = 0$ and $a_{j,o} = a_{0,l} = 0$. We infer $a_{i,j} = 0$ for all $0 \le j,l \le d-1$, and R is therefore reduced to $R = \sum_{0\le i,j,l\le d-1} a_{i,j,d-i,l} E^i K^j \otimes E^{n-i} K^l$.
Finally, we note that $\Delta^{op}(E^{d-1})$ must be different from zero, and then develop the expression $R\Delta(E^{d-1})$.
Writing $\Delta(E^{d-1}) = \sum_{0\le x,y,z\le d-1} b_{x,y,z} E^x K^y \otimes E^{d-1-x} K^z$ with $b_{x,y,z} \in \mathbb{C}$, we obtain $R\Delta(E^{d-1}) = \sum_{i,j,l,x,y,z} c_{i,j,l,x,y,z} E^{i+x} K^{j+y} \otimes E^{2d-i-x-1} K^{l+z}$.
Since either $i+x \ge d$ or $2d-i-x-1 \ge d$, we necessarily have $R\Delta(E^{d-1}) = 0$.
We thus arrive to the contradiction $R\Delta(E^{d-1})R^{-1} = 0$ and $R\Delta(E^{d-1})R^{-1} = \Delta^{op}(E^{d-1}) \neq 0$. $\square$

Remark 2.3 The case $n = 4$ yields a quasi-cocommutative Hopf algebra (see [2]). An alternative proof of Proposition 2.1 is provided in [2] using the presentation of u_q^+ by quiver and relations .

2.1 Modules.

The isomorphism classes of the modules described below constitute the complete list of isomorphism classes of indecomposable u_q^+-modules; they are all non-isomorphic. To each couple (i,u), where $i \in \mathbb{Z}/d\mathbb{Z}$ and $0 \le u \le d-1$, corresponds a u_q^+-module, denoted by M_i^u, of dimension $u+1$. It admits a basis $\{e_i^0, e_i^1, \dots, e_i^u\}$ over $\mathbb{C}$ such that the action of u_q^+ on the basis vectors is given by

$$\left\{ \begin{array}{lcl} Ke_i^j & = & q^{2(i+j)} e_i^j \\ Ee_i^j & = & e_i^{j+1} \ \text{ for } 0 \le j \le u-1 \\ Ee_i^u & = & 0 \end{array} \right.$$

Note that e_i^0 is a generator of M_i^u over u_q^+. The indecomposable projective modules are those of dimension $d-1$, and we denote them by $P_i = M_i^{d-1}$. The simple modules are the one-dimensional modules, and we denote them by $S_i = M_i^0$.

Notations : The length of a vector v belonging to a u_q^+-module is an integer $0 \leq m \leq d-1$, minimal for the property $E^{m+1}v = 0$. In particular, the length of a basis vector of the type e_i^j is $u-j$. For $r, s \in \mathbb{Z}$ let $\mathrm{E}(r/s)$ be the entire part of r/s.

3 Axiomatisation of the tensor product of modules.

The tensor product of modules on u_q^+ has decomposition formulas which are similar to those for the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, and for the quantum universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ when q is not a root of unity. The following axiomatisation unifies the proofs of these formulas leaving behind the concrete decomposition.

Remark 3.1 Recall that the Grothendieck group of a ring Λ, denoted by $K(\Lambda)$, is the quotient of the free abelian group with basis the isomorphism classes $[X]$ of modules X on Λ by the subgroup generated by elements $[X_2] - [X_1] - [X_3]$ provided by each split exact sequence $X_1 \to X_2 \to X_3$ of Λ-modules. Moreover if Λ is a Hopf algebra, the free abelian group is endowed with a ring structure through the tensor product of modules. The functor induced by tensoring over the ground field is flat, implying that the subgroup above is an ideal, and hence the quotient $K(\Lambda)$ is still a ring.
If Λ is a finite dimensional algebra, its Grothendieck group is a free abelian group with basis given by the isomorphism classes of indecomposable modules.

Let I be the set $\{0\}$ or $\mathbb{Z}/d\mathbb{Z}$. To m belonging to $\overline{\mathbb{N}} - \{0\}$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \infty$, put J_m to be the set $\{0, \ldots, m-1\}$ if $m \in \mathbb{N}$ and $J_m = \mathbb{N}$ if $m = \infty$. Consider the free commutative group generated by the elements [i,u], where (i,u) belong to $I \times J_m$. Suppose now that this group is equipped with an extra multiplicative structure, making it into a ring. Denote by $\oplus$ the addition law and $\otimes$ the multiplication law. We need to put $[i,u] = 0$ if $u < 0$. We have the proposition :

Proposition 3.1 *Assume the relations below hold and are symmetric with respect to* $\otimes$ *:*
$[i,0] \otimes [j,0] = [i+j,0]$, $[0,1] \otimes [j,v] = [j,v+1] \oplus [j+1,v-1]$ *for* $0 \leq v \leq m-2$ *,*
and $[0,1] \otimes [j,m-1] = [j,m-1] \oplus [j+1,m-1]$ *where* $(i,j) \in I \times J_m$ *and* $u, v \in J_m$.
Then the following decomposition formulas are true:

1. $[i,u] \otimes [j,v] = \oplus_{l=0}^{min(u,v)} [i+j+l, u+v-2l]$ *for* $u+v \leq m-1$
2. $[i,u] \otimes [j,v] = \oplus_{l=0}^{e} [i+j+l, m-1] \oplus \oplus_{l=e+1}^{min(u,v)} [i+j+l, u+v-2l]$ *for* $u+v \geq m-1$ *where* $e = u+v-(m-1)$

Proof : We proceed by double induction. First we prove that $[i,u]\otimes[j,0]=[i+j,u]$ for all $i,j\in I$ and $u\leq m-1$ by induction on u. By assumption it is true for $u=0$. Suppose it is valid up to a rank $0<u<m-1$ and let's show it for $u+1$. For this purpose we look at $[0,1]\otimes[i,u]\otimes[j,0]$. Developing the left and right side respectively we obtain the equality :
$([i,u+1]\otimes[j,0])\oplus([i+1,u-1]\otimes[j,0])=[0,1]\otimes[i+j,u]$
that is $([i,u+1]\otimes[j,0])\oplus[i+j+1,u-1]=[i+j,u+1]\oplus[i+j+1,u-1]$, and as a consequence $[i,u+1]\otimes[j,0]=[i+j,u+1]$.
Next, we take an arbitrary u, and show the formulas by induction on v. Suppose they hold up to a rank $v\geq 1$, then we have two situations to consider, either $u+v+1\leq m-1$ or $u+v+1\geq m-1$. Developing $[0,1]\otimes[i,u]\otimes[j,v]$ on the left and right hand side respectively easily solves the first case. For the second more care is needed. Set $e=u+v-(m-1)$ and let $\oplus_{l=0}^{a}[x_l,y_l]=0$ if $a\leq 0$ with $(x_l,y_l\in I\times J_m)$. We proceed as before by developing the left and right sides of $[i,u]\otimes[j,v]\otimes[0,1]$ and thus obtaining the equality
$(\oplus_{l=0}^{e}[i+j+l,m-1]\oplus\ \oplus_{l=e+1}^{min(u,v)}[i+j+l,u+v-2l])\otimes[0,1]=$
$[i,u]\otimes([j,v+1]\oplus[j+1,v-1])$. Developing this gives us the identity
$(\oplus_{l=0}^{e}([i+j+l,m-1]\oplus[i+j+l+1,m-1])\oplus$
$\oplus_{l=e+1}^{min(u,v)}([i+j+l,u+v-2l+1]\oplus[i+j+l+1,u+v-2l-1])=$
$[i,u]\otimes[j,v+1]\oplus\oplus_{l=0}^{e-1}[i+j+1+l,m-1]\oplus\ \oplus_{l=e}^{min(u,v-1)}[i+j+1+l,u+v-1-2l])$.
Therefore $[i,u]\otimes[j,v+1]=\oplus_{l=0}^{e+1}[i+j+l,m-1]\oplus\oplus_{l=e+2}^{min(u,v+1)}[i+j+l,u+v+1-2l]$. $\square$

Remark 3.2 The Grothendieck ring of the Hopf algebra u_q^+ corresponds to $I=\mathbb{Z}/d\mathbb{Z}$ and $m=d$ where we replace the formal writing $[i,u]$ by the isomorphism class of the indecomposable module $[M_i^u]$. This observation leads us to the next result.

Theorem 3.1 *Let M_i^u and M_j^v be indecomposable u_q^+-modules for $i,j\in\mathbb{Z}/d\mathbb{Z}$ and $0\leq u,v\leq d-1$. There are isomorphisms :*

1. *If $u+v\leq d-1$*

$$M_i^u\otimes M_j^v\cong\bigoplus_{l=0}^{min(u,v)}M_{i+j+l}^{u+v-2l}$$

2. *If $u+v\geq d-1$, set $e=u+v-(d-1)$, then*

$$M_i^u\otimes M_j^v\cong\bigoplus_{l=0}^{e}P_{i+j+l}\oplus\bigoplus_{l=e+1}^{min(u,v)}M_{i+j+l}^{u+v-2l}$$

Proof : In view of the previous remark we can apply the proposition. We need to check that $S_i \otimes S_j \cong S_{i+j}$ and $M_i^1 \otimes S_j \cong M_{i+j}^1$ as well as $M_i^1 \otimes M_j^v \cong M_j^v \otimes M_i^1 \cong M_{i+j}^{v+1} \oplus M_{i+j+1}^{v-1}$ and finally that $M_i^1 \otimes P_j \cong P_j \otimes M_i^1 \cong P_{i+j} \oplus P_{i+j+1}$.
The first two isomorphisms are simply given by letting $e_i^0 \otimes e_j^0$ go to a non zero multiple of e_{i+j}^0.
To prove the third assertion (we treat the case $M_i^1 \otimes M_j^v \cong M_{i+j}^{v+1} \oplus M_{i+j+1}^{v-1}$) we need to ensure that in $M_i^1 \otimes M_j^v$ we have two vectors w_1 and w_2, of lengths $v+1$ and $v-1$ respectively, and whose $K-$eigenvalues are respectively $q^{2(i+j)}$ and $q^{2(i+j+1)}$. Indeed this implies the existence of M_{i+j}^{v+1} and M_{i+j+1}^{v-1} as submodules of $M_i^1 \otimes M_j^v$, as well as their sum which is necessarily direct. For dimension reasons we therefore obtain the required isomorphism.
Let us make explicit the vectors w_1 and w_2. For w_1 we simply take $e_i^0 \otimes e_j^0$. What needs to be checked is that $E^{v+1} e_i^0 \otimes e_j^0 \neq 0$ (note that $E^{v+2} e_i^0 \otimes e_j^0$ is necessarily equal to 0). Using the comultiplication formulas given in section 2 we find that $E^{v+1} e_i^0 \otimes e_j^0 = q^{-v} \begin{bmatrix} v+1 \\ 1 \end{bmatrix}_q q^{2(v+j)} e_i^0 \otimes e_j^v$; this is equal to $\frac{q^{v+1}-q^{-v-1}}{q-q^{-1}} e_i^1 \otimes e_j^v$ which is not equal to 0 since we are in the case $v \leq d-1$. To determine w_2 we need to make two computations : first, let a, b belong to k, then we have $E^{v-1}(a e_i^1 \otimes e_j^0 + e_i^0 \otimes e_j^1) = b e_i^0 \otimes e_j^v + (a + b(q^{2j+v-2}) \frac{q^{v-1}-q^{-(v-1)}}{q-q^{-1}}) e_i^1 \otimes e_j^{v-1}$, which is non-zero whenever a and b are both different from zero. Next, we compute $E^v(a e_i^1 \otimes e_j^0 + b e_i^0 \otimes e_j^1)$ and find it to be equal to $a e_i^1 \otimes e_j^v + q^{2j+v+1} \frac{q^v - q^{-v}}{q-q^{-1}} b e_i^1 \otimes e_j^v$. In view of these computations, we set $w_2 = a e_i^1 \otimes e_j^0 + b e_i^0 \otimes e_j^1$, with $a = -q^{2(j+v)} + 1 + q^{2j+1}$ and $b = q - q^{-1}$, and hence obtain a vector satisfying the desired conditions.
$\square$

Remark 3.3 We will see that the theorem can be obtained in a totally different way, by means of extendable u_q^+-modules.

Next we consider two different cases where our axiomatisation applies.

Proposition 3.2 *Taking $I = 0$ and $m = \infty$ leads to Clebsch-Gordan formulas for $U(\mathfrak{sl}_2(\mathbb{C}))$ and $U_q(\mathfrak{sl}_2(\mathbb{C}))$ when q is not a root of unity.*

Proof : 1) Recall the irreducible representations of $U(\mathfrak{sl}_2(\mathbb{C}))$. To each integer n corresponds a simple $U(\mathfrak{sl}_2(\mathbb{C}))$-module $V(n)$ of dimension $n+1$. It admits a basis $\{v_0, \dots, v_n\}$ over k such that the action of $U(\mathfrak{sl}_2(\mathbb{C}))$ is given by

$$\begin{cases} X v_i &= (n-i+1) v_{i-1} \\ Y v_i &= (i+1) v_{i+1} \\ H v_i &= (n-2i) v_i \end{cases} \quad \text{where } v_i = 0 \text{ for } i \notin \{0, \dots, n\}$$

and we have the Clebsch-Gordan formula for the decomposition of the tensor product of two such modules : $V(n)\otimes V(m)\cong\oplus_{l=0}^{min(n,m)}V(n+m-2l)$. In view of the preceding results, this formula can be obtained by checking the following isomorphisms of $U(\mathfrak{sl}_2(\mathbb{C}))$: $V(0)\otimes V(0)\cong V(0)$ and
$V(1)\otimes V(n)\cong V(n+1)\oplus V(n-1)$ for $n\geq 1$.
The first is trivial, the second is obtained by giving an explicit decomposition as it was done for u_q^+. Indeed, let $\{v_0, v_1\}$ and $\{v'_0,\dots,v'_n\}$ be the basis of $V(1)$ and $V(n)$ respectively. Then the vectors $v_0\otimes v'_0$ and $v_0\otimes v'_1-mv_1\otimes v'_0$ are generators of the modules $V(n+1)$ and $V(n-1)$ respectively. Their sum is a direct sum and comparing the dimensions leads to the desired isomorphism.
2) The case of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ when q is not a root of unity is similar. Let $\epsilon=\pm 1$. To each integer n correspond two modules $V_{1,n}$ and $V_{-1,n}$ who admit bases $\{v_{\epsilon,0},\ v_{\epsilon,1},\dots,v_{\epsilon,n-1}\}$ such that the action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is given by

$$\begin{cases} Ev_{\epsilon,i} &= \epsilon[n-i+1]v_{\epsilon,i-1} \\ Fv_{\epsilon,i} &= \epsilon[i+1]v_{\epsilon,i+1} \qquad \text{where } v_{\epsilon,i}=0 \text{ for } i\notin\{0,\dots,n\} \\ Kv_{\epsilon,i} &= \epsilon q^{n-2i}v_{\epsilon,i}. \end{cases}$$

The Clebsch-Gordan formula is : $V_{\epsilon,n}\otimes V_{\epsilon',m}\cong\oplus_{l=0}^{min(n,m)}V_{\epsilon\epsilon',n+m-2l}$. One easily reduces to the case of modules of type $V_{1,n}$ and as in the former situations the isomorphism between $V_{1,1}\otimes V_{1,n}$ and $V_{1,n+1}\oplus V_{1,n-1}$ for $n\geq 1$ is guaranteed by the two vectors $v_0\otimes v'_0$ and $v_0\otimes v'_1-[m]q^{-m}v_1\otimes v'_0$ (we assume that the vectors v_i and v'_j form bases for $V_{1,1}$ and $V_{1,n}$ respectively).
$\square$

Remark 3.4 Considering the simple u_q^+-modules, we can observe that they form a multiplicative group for the tensor product, isomorphic to the cyclic group of order n. Actually, the isomorphism classes of simple modules over a basic and split Hopf algebra always provide a group (see for instance [4]). Now this group acts on the category of u_q^+-modules via the tensor product and it is interesting to note that the action of the generator S_1 on an indecomposable module yields the dual transpose (see [1]).

4 Extendable modules.

It is obvious that a u_q^+-module is not in general issued from a u_q-module, in the sense that it is not obtained by restricting the action of u_q to u_q^+. Nevertheless we can consider the subfamily of u_q^+-modules on which indeed there exists an action of u_q such that the original action of u_q^+ is respected. We call those modules extendable. They have the property that the R-matrix of u_q provides isomorphisms making the tensor product of two such modules commutative. Restricting our study to this family gives some information on the decomposition of u_q^+-modules, as well as on simple u_q-modules. We

need the following notation :
Notation : Let $u \in \mathbb{N}$, then $\overline{u}$ is the representative element of the class of u modulo d contained in the set $\{0, \dots, d-1\}$.

Theorem 4.1 *The extendable indecomposable modules are :*

1. *The indecomposable modules of type $M_i^{\overline{-2i}}$ for $0 \leq i \leq d-1$. These modules extend in a unique way and provide all the simple* u_q*-modules.*

2. *The projective indecomposable modules P_i for $0 \leq i \leq d-1$. These modules extend in two non-isomorphic ways, except $P_{\frac{d+1}{2}}$ when d is odd.*

Proof : We proceed in the following way : First we consider an arbitrary indecomposable u_q^+-module, and we try to define an action of $F \in \mathrm{u}_q$ on its basis elements, such that the original action of u_q^+ is preserved, and the algebra structure of u_q is respected. We thus infer the necessary conditions for an indecomposable module to be extendable.
Consider a module M_i^u with $i \in \mathbb{Z}/d\mathbb{Z}$ and $0 \leq u \leq d-1$. It is generated over u_q^+ by the element e_i^0, and the set $\{e_i^j\}_{0 \leq j \leq u}$ is a basis over k. The action of u_q^+ is given by $Ee_i^j = e_i^{j+1}$ for $0 \leq j \leq u-1$, $Ee_i^u = 0$ and $Ke_i^j = q^{2(i+j)}e_i^j$.
Suppose we have an action of F given by $Fe_i^j = \sum_{o \leq h \leq u} \lambda_{i,j}^h e_i^h$ where $\lambda_{i,j}^h \in \mathbb{C}$. The relation $KF = q^{-2}FK$ implies $KFe_i^j = \sum_{0 \leq h \leq u} \lambda_{i,j}^h q^{2(i+h)} e_i^h$ $= q^{-2}FKe_i^j = q^{-2}q^{2(i+j)} \sum_{0 \leq h \leq u} \lambda_{i,j}^h e_i^h$.
It follows that $\lambda_{i,j}^h q^{2(i+h)} = q^{2(i+j-1)} \lambda_{i,j}^h$ and therefore :
$Fe_i^j = \lambda_{i,j}^{j-1} e_i^{j-1} = \lambda_i^{j-1} e_i^{j-1}$ for all $1 \leq j \leq u$ and $Fe_i^0 = \lambda_i^{d-1} e_i^{d-1}$,
where $\lambda_i^{d-1} = 0$ if $u \leq d-2$. Since $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$, we must have the following :

$$\lambda_i^0 = -[2i] + \lambda_i^{d-1}$$

We next proceed by induction and obtain

$$\lambda_j = \sum_{0 \leq h \leq j} -[2(i+h)] + \lambda_i^{d-1}.$$

The remaining relations are now $Ee_i^u = 0$ and $F^d = 0$. From the first one we deduce :

$$(EF - FE)e_i^u = E\lambda_i^{u-1} e_i^{u-1} = \lambda_i^{u-1} e_i^u = \frac{K - K^{-1}}{q - q^{-1}} e_i^u = \frac{q^{2(i+u)} - q^{-2(i+u)}}{q - q^{-1}} e_i^u.$$

On the other hand, $\lambda_{u-1} = \sum_{0 \leq h \leq u-1} -[2(i+h)] + \lambda_i^{d-1}$. The equality is automatically realized when dealing with a projective module. Otherwise, that is when $u \leq d-2$, we need

$$
\begin{aligned}
&\sum_{0\leq h\leq u} \frac{q^{-2(i+h)}-q^{2(i+h)}}{q-q^{-1}}\\
= \quad & \frac{-q^{2i}}{q-q^{-1}}\left(\frac{1-q^{2(u+1)}}{1-q^{2}}\right)+\frac{q^{-2i}}{q-q^{-1}}\left(\frac{1-q^{-2(u+1)}}{1-q^{-2}}\right)\\
= \quad & \frac{q^{2i}(1-q^{2(u+1)}+q^{-4i+2}-q^{-2(u+1)+2-4i})}{(q-q^{-1})(q^{2}-1)}\\
= \quad & \frac{q^{2i}(1-q^{2(u+1)})(1-q^{-2u-4i})}{(q-q^{-1})(q^{2}-1)}=0
\end{aligned}
$$

The equality is true when $2(u+1)=0 \mod n$ and $2u=-4i \mod n$. For n odd the first case is never realized, and for n even it corresponds to the projective modules. Otherwise we need the condition $u=-2i \mod d$.
The last condition on the λ_i^j coming from $F^d=0$ is $\lambda^{d-1}_{\frac{d+1}{2}}=0$ for d odd. Hence the indecomposable modules for which the action of u_q^+ extends to u_q are the projectives and the modules of the type $M_i^{\overline{-2i}}$. For $i\in\{0,\dots,d-1\}$, it is easy to check that the modules obtained on u_q from the modules $M_i^{\overline{-2i}}$ are simple, and we thus obtain all the simple modules on u_q up to isomorphism (the list of simple u_q-modules is given in [5]).□

Remark 4.1 The projective modules are examples of modules extendable to u_q-modules in two non-isomorphic ways. We are therefore allowed to imagine the case of an extendable module whose indecomposable components are not extendable. This turns out to be impossible.

Proposition 4.1 *A u_q^+-module is extendable if and only if it is a direct sum of indecomposable extendable modules.*

Proof : Let X be an arbitrary u_q^+-module, decomposable into $M_i^u\oplus(\bigoplus_{l\in L\ v\in V} M_l^v)$, where L is a finite set. We examine the possible actions of F on the basis $\{e_i^j\}$ of M_i^u. Using a simple induction and the relation $EF-FE=\frac{K-K^{-1}}{q-q^{-1}}$, we find that an action must be of the form :
$F.e_i^j=\lambda_i^{j-1}e_i^{j-1}+$ linear combination of $\{e_l^k\}_{j\leq k}$.
The action of E on e_i^u given by $Ee_i^u=0$ requires that

$$
\begin{aligned}
(EF-FE)e_i^u \quad &= \frac{q^{2(i+u)}-q^{-2(i+u)}}{q-q^{-1}}e_i^u\\
&= EF(e_i^u)\\
&= (\lambda_i^u e_i^u+\text{lin.comb.}\{e_l^k\}_{u+1\leq k})
\end{aligned}
$$

This implies that $\lambda_i^{u-1}=\frac{q^{2(i+u)}-q^{-2(i+u)}}{q-q^{-1}}$, i.e. that M_i^u is an extendable module. □

Remark 4.2 There may be more than one way to extend a direct sum of non projective, indecomposable, extendable modules. As an example we can give the u_q^+-module $M_1^1\oplus S_0$ in the case $n=3$. Indeed the possible

actions of F are easily found to be : $Fe_1^0 = ce_0^0$, $Fe_1^1 = \lambda_1^0 e_1^0$ and $Fe_0^0 = 0$ where c belongs to $\mathbb{C}$. Considering the options $c = 0$ and $c \neq 0$ respectively, the result is two non-isomorphic representations of u_q.

Remark 4.3 - For d odd there is exactly one indecomposable extendable module per dimension m, where $1 \leq m \leq d-1$.
- For d even there are exactly two indecomposable extendable modules per dimension $2m+1$, where $0 \leq m \leq \frac{d}{2} - 1$.

The following result provides a characterisation of self-dual indecomposable modules in terms of extendable ones. We recall that the dual $Hom_k(M,k)$ of a module M over a Hopf algebra H over a field k can be provided with a left H-module structure by means of the antipode S (see [10]) (we denote this left H-module by ${}^*M$) :

$\lambda.f(x) = f(S(\lambda)x)$ for $\lambda \in H$, $f \in Hom_k(M,k)$ and $x \in M$.

Proposition 4.2 *Let M be a u_q^+-module. Then the following are equivalent*

1. *The module M is indecomposable and self-dual.*
2. *The module M is indecomposable and extendable of type M_i^u with $u \equiv -2i$.*

Proof : We consider an arbitrary indecomposable module M_i^u. Let $\{(e_i^j)^*\}$ be the dual basis of ${}^*M_i^u$; then $(e_i^u)^*$ is a generator of this module and we have another basis given by the elements $\{E^j(e_i^u)^*\}_{0\leq j\leq u-1}$. The action of K on $E^j(e_i^u)^*$ is the following :
$KE^j(e_i^u)^* = q^{2j}E^jK(e_i^u)^* = q^{2(j-i-u)}E^j(e_i^u)^*$.
We deduce an isomorphism between ${}^*M_i^u$ and M_{n-i-u}^u. The explicit isomorphism is
$M_i^u \longrightarrow^* M_{n-i-u}^u$
$e_i^j \longmapsto (-1)^j q^{j(j+2i+1)} e_{n-i-u}^*$
Consequently M_i^u is selfdual iff $u \equiv -2i$. $\square$

Remark 4.4 The extendable modules provide a different proof of the Clebsch-Gordan-like formula for u_q^+ stated before. We sketch the proof briefly.

Proof : The first step does not involve the extendable modules (see [2] where the proof is incomplete). It consists in showing that the tensor product of two arbitrary indecomposable u_q^+-modules must decompose as follows : $M_i^u \otimes M_j^v \cong \oplus_{l=0}^v M_{i+j+l}^{x_l}$ where $u - d \leq x_l \leq u+v-d$ and $u+v \leq d-1$ (we request the latter condition here in order to simplify, and we suppose that $v \leq u$). This is done by considering the dimension of each K-eigenspace and the action of E on those. Indeed the K-eigenvalues are $q^{2(i+j+l)}$ with $0 \leq l \leq u+v$, and the dimensions are distributed as follows : to $q^{2(i+j+l)}$ with $0 \leq l \leq v$ corresponds a vector space of dimension $l+1$, moreover the

vector space morphism induced by E between the eigenspace of eigenvalue $q^{2(i+j+l)}$ and the one of eigenvalue $q^{2(i+j+l+1)}$ is injective. To the same situation with $v \leq l \leq u$ corresponds a vector space of dimension $v+1$ and the morphism induced by E is one to one. Finally, for $u \leq l \leq u+v$ the dimension is $u+v-l+1$, and E induces a surjective morphism whose kernel is one-dimensional. As a consequence the quotient by the action of the Jacobson radical, $\mathrm{top}(M_i^u \otimes M_j^v)$, is $\oplus_{l=0}^{min(u,v)} S_{i+j+l}$ and we conclude by uniseriality.
Now in the specific case of two indecomposable and extendable u_q^+-modules, necessarily $x_l = u+v-2l$, which is the result we want in the general case. Indeed, to each index $i+j+l$ corresponds one and only one extendable indecomposable module. Moreover the tensor product of two extendable modules is still extendable, hence it decomposes into a direct sum of indecomposable extendable modules, and leaves only one choice for the value of x_l. Denote by ϕ the resulting isomorphism.
This observation on the extendable modules immediately leads to the solution of the general case. Let X and Y be the indecomposable extendable modules of dimension $u+1$ and $v+1$ respectively, and let S be the simple module s.t. $M_i^u \otimes M_j^v \cong S \otimes X \otimes Y$. Then the morphism $id \otimes \phi$ realizes the required decomposition isomorphism. □

Remark 4.5 The R-matrix of u_q provides isomorphisms through the action of τR between $M_i^u \otimes M_j^v$ and $M_j^v \otimes M_i^u$ when these are extendable modules. For any simple module S_l, induced isomorphisms are given between $S_l \otimes M_i^u \otimes M_j^v$ and $S_l \otimes M_j^v \otimes M_i^u$ by $id_{S_l} \otimes \tau R$. Hence explicit isomorphisms are obtained, which make the tensor product of any two modules lying in the orbit of the extendable modules under the action of the structure group commutative (see remark 3.4). We let Indu_q^+ denote the set of indecomposable finite dimensional u_q^+-modules, and we have the following corollary.

Corollary 4.1 *1. When d is odd, the orbit, under the action of the structure group, of the extendable indecomposables is* Indu_q^+*, hence isomorphisms are obtained in all cases.*

2. When d is even the orbit covers all the indecomposables whose dimension over k is odd. Hence isomorphisms are given between $M_i^u \otimes M_j^v$ and $M_j^v \otimes M_i^u$ when u and v are even.

The explicit isomorphisms obtained when d is odd are not natural, since u_q^+ is not quasi-cocommutative. Nevertheless they satisfy the other relations defining a braided module category (see [5]). Denote by $c_{U,V}$ the isomorphism between $U \otimes V$ and $V \otimes U$, where U, V are u_q^+-modules. Then we have the following :

Corollary 4.2

$$c_{U,V\otimes W} = (id_V \otimes c_{U,W})(c_{U,V} \otimes id_W)$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes id_V)(id_U \otimes c_{V,W})$$

$$(id_W \otimes c_{U,V})(c_{U,W} \otimes id_V)(id_U \otimes c_{V,W}) = (c_{V,W} \otimes id_U)(id_V \otimes c_{U,W})(c_{U,W} \otimes id_W)$$

Proof : We show the first equality, the others are obtained in a similar way. There exist extendable modules M_1, M_2 and M_3 together with a simple module S and isomorphisms :
$\phi_1 \ : \ U \otimes V \otimes W \cong S \otimes M_1 \otimes M_2 \otimes M_3$
$\phi_2 \ : \ V \otimes W \otimes U \cong S \otimes M_2 \otimes M_3 \otimes M_1$
$\phi_3 \ : \ V \otimes U \otimes W \cong S \otimes M_2 \otimes M_1 \otimes M_3.$
Then

$$\begin{aligned} c_{U,V\otimes W} &= \phi_2^{-1}(id_S \otimes c_{M_1,M_2\otimes M_3})\phi_1 \\ &= \phi_2^{-1}(id_S \otimes (id_{M_2} \otimes c_{M_1,M_3}) \circ id_S \otimes (c_{M_1,M_2} \otimes id_{M_3}))\phi_1 \\ &= \phi_2^{-1}((\phi_2 \circ id_V \otimes c_{U,V} \circ \phi_3^{-1}) \circ (\phi_3 \circ c_{U,V} \otimes id_W \circ \phi_1^{-1}))\phi_1 \\ &= (id_V \otimes c_{U,W})((c_{U,V} \otimes id_W). \ \Box \end{aligned}$$

Remark 4.6 The underlying isomorphism of vector spaces $M_i^u \otimes M_j^v \cong M_j^v \otimes M_j^v$ does not depend on i and j, therefore we obtain no new solution to the Yang-Baxter equation.

5 Tensor product of simple u_q-modules.

Recall that the simple u_q-modules are obtained from indecomposable extendable u_q^+-modules (see proposition 4.1). We denote by $\overline{M}_i^u$, where $i \in \mathbb{Z}/n\mathbb{Z}$ and $0 \leq u \leq d-1$, a simple module over u_q. We need to recall a family of indecomposable finite dimensional u_q-modules, which are both projective and injective (see [8] and [9]). To begin with, take the direct sum of the projective indecomposable u_q^+-modules $P_i \oplus P_{\overline{-2i}}$, where $i \in \{0, \ldots, \mathrm{E}((d-1)/2)\}$. Then we define the following action of F on its basis elements, making it into a u_q-module : $Fe_i^j = \lambda_i^{j-1} e_i^{j-1}$ and $Fe^j_{\overline{-2i}} = e_i^{\overline{-2i+j}} + \lambda^{j-1}_{\overline{-2i}} e^{j-1}_{\overline{-2i}}$ where $j \in \{0, \ldots, \overline{4i-1}\}$ and $Fe^j_{\overline{-2i}} = \lambda^{j-1}_{\overline{-2i}} e^{j-1}_{\overline{-2i}}$ for $j \in \{\overline{4i-1}+1, \ldots, d-1\}$. We denote the resulting modules by $\tilde{P}_i$. In [8] Reshetikhin and Turaev give decomposition formulas for the tensor product of simple u_q-modules. The proof is based on the study of indecomposable modules on u_q ; the Verma modules and autoinjective modules as well as exact sequences of these. These decomposition formulas are established here by a totally different approach, using the preceding results obtained on u_q^+-modules.

Theorem 5.1 *Let $\overline{M}_i^u$ and $\overline{M}_j^v$ be simple u_q-modules for $i, j \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq u, v \leq d-1$ and $u+v \leq d-1$. Suppose $v \leq u$. There is an isomorphism*

$$\overline{M}_i^u \otimes \overline{M}_j^v \cong \bigoplus_{l=0}^{v} \overline{M}_{i+j+l}^{u+v-2l}.$$

Proof : We simply show that there's a unique way extending the direct sum $X = \oplus_{l=0}^{v} M_{i+j+l}^{u+v-2l}$, that is by extending each module separately.
Recall that M_{i+j+l}^{u+v-2l} is generated by e_{i+j+l}^{0} as a u_q^+-module and admits the set $\{e_{i+j+l}^{k}\}_{0\leq k\leq u+v-2l}$ as a basis over k.
Recall also that the unique extended action of u_q on M_{i+j+l}^{u+v-2l} is given by $Fe_{i+j+l}^{k} = \lambda_{i+j+l}^{k-1} e_{i+j+l}^{k-1}$.

In order to extend X, we study the possible actions of F on the basis elements. They are entirely determined by the action of F on the generators of each indecomposable module. Indeed, $Fe_{i+j+l}^{k} = \lambda_{i+j+l}^{k-1} e_{i+j+l}^{k-1} + E^k(Fe_{i+j+l}^{0})$. Let us first show that Fe_{i+j+l}^{0} is necessarily a linear combination of elements of the set $\{e_{i+j+l-k}^{k-1}\}_{1\leq k\leq l-1}$. Suppose Fe_{i+j+l}^{0} is a linear combination of elements $e_{i+j+k}^{m_k}$ with $0 \leq k \leq v$ and $0 \leq m_k \leq u+v-2k$. Applying the identity $KF = q^{-2}FK$, we find that m_k is congruent to $l-k-1$ modulo d. Therefore $m_k = l-k-1+pd$ with $p \in \mathbb{Z}$. Since $0 \leq m_k \leq u+v-2k$, necessarily $p = 0$ and $m_k = l-k-1$. Consequently we can write $Fe_{i+j+l}^{0} = \sum_{k=1}^{l-1} a_k e_{i+j+l-k}^{k-1}$ with $a_k \in k$. Using the relation $EF - FE = \dfrac{K-K^{-1}}{q-q^{-1}}$, our previous observation on the action of F on an arbitrary basis element implies

$$Fe_{i+j+l}^{m} = \lambda_{i+j+l}^{m-1} e_{i+j+l}^{m-1} + \sum_{k=1}^{l-1} a_k e_{i+j+l-k}^{k-1+m}.$$

Finally, since $Ee_{i+j+l}^{u+v-2l} = 0$, we must have that $EFe_{i+j+l}^{u+v-2l} = \lambda_{i+j+l}^{u+v-2l-1} e_{i+j+l} u+v-2l-1$. This implies

$\displaystyle\sum_{k=1}^{l-1} e_{i+j+l-k}^{u+v-2l+k} = 0$. But for $0 \leq m \leq u+v-2(l-k)$ we have $e_{i+j+l-k}^{m} \neq 0$, and since $0 \leq u+v-2l+k \leq u+v-2l+2k$, we find that $a_k = 0$ for $1 \leq k \leq l-1$. Hence $Fe_{i+j+l} = 0$ and $Fe_{i+j+l}^{m} = \lambda_{i+j+l}^{m-1} e_{i+j+l}^{l-1}$. □

Theorem 5.2 *Let $\overline{M}_i^u$ and $\overline{M}_j^v$ be simple u_q-modules for $i,j \in \mathbb{Z}/d\mathbb{Z}$, $0 \leq u,v \leq d-1$ and $u+v \geq d-1$. There is an isomorphism*

$$\overline{M_i^u} \otimes \overline{M_j^v} \cong \bigoplus_{l=0}^{\mathrm{E}(e/2)} \tilde{P}_{i+j+l} \oplus \bigoplus_{l=e+1}^{min(u,v)} \overline{M}_{i+j+l}^{u+v-2l}$$

Proof : We can observe three cases :

$$\begin{cases} u = & d-2i \\ v = & d-2j \end{cases}, \quad \begin{cases} u = & 2d-2i \\ v = & d-2j \end{cases} \text{ and } \begin{cases} u = & 2d-2i \\ v = & 2d-2j. \end{cases}$$

We restrict ourselves to the first case since the only difference between these is of elementary computational order. We furthermore assume that $\min(u,v) = v$.

Step 5.2.1 *The tensor product decomposes in the following sum :* $\overline{M_i^u} \otimes \overline{M_j^v} \cong \overline{\oplus_{l=0}^e P_{i+j+l}} \oplus \oplus_{l=e+1}^v \overline{M}_{i+j+l}^{u+v-2l}$.

Proof : As in the preceding proof, Fe_{i+j+k}^0 is a linear combination of K-eigenvectors with K-eigenvalue equal to $q^{2(i+j+k-1)}$, and $Fe_{i+j+k}^l = \lambda_{i+j+k}^{l-1} e_{i+j+k}^{l-1} + E^l F e_{i+j+k}^0$. First we consider the decomposition as a u_q^+-modules decomposition and show that the element Fe_{i+j+l}^0 is not in $\oplus_{k=e+1}^v M_{i+j+k}^{u+v-2k}$ for $0 \leq l \leq e$. Indeed, for $1 \leq l \leq e$ the K-eigenvalue of the vector Fe_{i+j+l}^0 is $q^{2(i+j+l-1)}$ (note that $Fe_{i+j}^0 = 0$), whereas for $e+1 \leq k \leq v$ and $0 \leq m_k \leq u+v-2k$ the K-eigenvalue for the vector $e_{i+j+k}^{m_k}$ is $q^{2(i+j+k+m_k)}$. Asking $2(i+j+k+m_k)$ to be congruent to $2(i+j+l-1)$ modulo n is equivalent to require that $k+m_k \equiv l-1 \mod d$. But $k+m_k \in \{e+1, \dots, d-2\}$ and $l-1 \in \{0, \dots, e-1\}$, therefore this congruence is impossible. On the other hand, a computation similar to that of the proof of the preceding proposition shows that $Fe_{i+j+e+l}^0 = 0$ for $l = 1, \dots, v$. Hence the first step.

Step 5.2.2 *There exists a* u_q^+*-decomposition of* $\overline{M_i^u} \otimes \overline{M_j^v}$ *such that for* $0 \leq k \leq \mathrm{E}(e/2)$*, the action of* F *on the generators* e_{i+j+k}^0 *of the* u_q^+*-modules* P_{i+j+k} *is zero.*

Proof : We show that there exists a K-eigenvector with eigenvalue $q^{2(i+j+k)}$ (unique up to scalar multiples) for $0 \leq k \leq e$, s.t. F acts on this vector as zero. Furthermore, we show that for $0 \leq k \leq \mathrm{E}(e/2)$, this vector is of length $d-1$ and hence generates a projective u_q^+-module.
The list of basis-vectors with K-eigenvalue equal to $q^{2(i+j+k)}$ is given by the following set of $e+1$ vectors :
$\{e_i^0 \otimes e_j^k \ ,\ e_i^1 \otimes e_j^{k-1}, \dots, e_i^k \otimes e_j^0 \ , e_i^u \otimes e_j^{d+k-u} \ ,\ e_i^{u-1} \otimes e_j^{d+k-u+1}, \dots ,$
$e_i^{u-(e-k-1)} \otimes e_j^v\}$.
The action of F induces a vector space morphism between the vector space generated by the above vectors and the vector space generated by the $e+1$ vectors of K-eigenvalue $q^{2(i+j+k-1)}$. The action of F is described by
$Fe_i^m \otimes e_j^{k-m} = q^{-2(i+m)} \lambda_j^{k-m-1} e_i^m \otimes e_j^{k-m-1} + \lambda_i^{m-1} e_i^{m-1} \otimes e_j^{k-m}$ and
$Fe_i^{u-m} \otimes e_j^{d+k-u+m} = q^{-2(u-m+i)} \lambda_j^{d+k-u+m-1} e_i^{u-m} \otimes e_j^{d+k-u+m-1}$
$+ \lambda_i^{u-m-1} e_i^{u+m-1} \otimes e_j^{d+k-u+m}$,
and the corresponding matrix has the following entries:

$$\begin{cases} a_{p,p} & = & q^{-2(i+p-1)} \lambda_j^{k-p} \neq 0 & \text{for } 1 \leq p \leq k-1 \\ a_{p,p+1} & = & \lambda_i^{p-1} \neq 0 & \text{for } 1 \leq p \leq k-1 \\ a_{p,p} & \neq & 0 & \text{for } k+2 \leq p \leq e+1 \\ a_{k+1,p} & = & 0 & \text{for } p \neq k+2 \\ 0 & & \text{otherwise.} & \end{cases}$$

We can make the following remarks : 1) The matrix is of rank e and consequently the kernel of the morphism is one-dimensional, which gives a unique

vector (up to scalar multiples), which we denote by v_k, s.t. $Fv_k = 0$.
2) This vector v_k is a linear combination of the basis vectors $e_i^m \otimes e_j^{k-1}$, which all appear with a non-zero coefficient. We can therefore put $v_k = e_i^k \otimes e_j^0 + w_k$ where w_k is a linear combination of $e_i^m \otimes e_j^{k-m}$ for $1 \leq m \leq k$.
3) The vectors $e_i^m \otimes e_j^{k-m-1}$ for $m \in \{0, \dots, k-1\}$ are all in the image of this morphism.
What remains to be satisfied is that $E^{d-1}v_k \neq 0$. For this purpose, we write v_k as above : $v_k = \ e_i^k \otimes e_j^0 + w_k$. Now there exists an integer m, between 0 and $d-1$, minimal for the property $E^{m+1}v_k = 0$. Consequently v_k generates an indecomposable u_q^+-module of the form M_{i+j+k}^m. Since $Fv_k = 0$, this u_q^+-module is an extendable indecomposable u_q^+-module, and so $m = d-1$ or m is congruent to $-2(i+j+k)$ mod d (thm. 4.1.). We need to exclude the second possibility. Suppose that m is congruent to $-2(i+j+k)$; this means that $m = d-2(i+j)-2k = e-2k-1$ for $\ 0 \leq k \leq \mathrm{E}((e-1)/2)$. If e is even and $k = e/2$, then $m = d-1$, and the two situations coincide. Observing that $u-k \geq e-k > e-2k-1$, we compute $E^{u-k}v_k = q^{2(u-k)}e_i^u \otimes e_j^0$ +(vectors linearly independant with $e_i^u \otimes e_j^0$). Necessarily $m > u-k$, which is a contradiction, and so $m = d-1$.
In P_{i+j+k} with $k \in \{0, \dots, \mathrm{E}((e-1)/2)\}$, we put $l_k = d-2(i+j+k)+1 = e-2k$, and we have $Fe_{i+j+k}^{l_k} = 0$ (see proof of theorem 4.1).

Step 5.2.3 *There exists a vector α_{l_k} such that $F\alpha_{l_k} = e_{i+j+k}^{l_k-1}$. Furthermore, the u_q^+-module generated by α_{l_k} is isomorphic to $P_{i+j+k+l_k}$.*

Proof : We observe that $l_k \in \{\mathrm{e}((e+1)/2), \dots, e\}$, and since e_{i+j+k}^0 is a linear combination of the vectors $e_i^0 \otimes e_j^k, \dots, e_i^k \otimes e_j^0$, we have that $e_{i+j+k}^{l_k-1}$ is a linear combination of the vectors $e_i^0 \otimes e_j^{l_k-1}, \dots, e_i^{l_k-1} \otimes e_j^0$. Therefore, considering the third remark in step 5.2.2, there exists a vector α_{l_k} with K-eigenvalue equal to $q^{2(i+j+k+l_k)}$ s.t. $F\alpha_{l_k} = e_{i+j+k}^{l_k-1}$. We now look at the u_q^+-module generated by α_{l_k}. There are two things to prove :
1) The module $\mathrm{u}_q^+\alpha_{l_k}$ is extendable. First of all, the sum $P_{i+j+k} + \mathrm{u}_q^+\alpha_{l_k}$ of u_q^+-modules is a direct sum. In order to prove this, we show that the vectors e_{i+j+k}^m and $E^m\alpha_{l_k}$ for $m \in \{0, \dots, d-1\}$ are linearly independant. Considering their K-eigenvalues, this means that we must have $E^{d-m}\alpha_{l_k} \neq a_m e_{i+j+k}^{l_k-m}$ and $E^s\alpha_{l_k} \neq a_s e_{i+j+k}^{l_k+s}$ for $m \in \{0, \dots, l_k-1\}$ and $s \in \{l_k, \dots, d-1\}$. Indeed, if we suppose $E^{d-m}\alpha_{l_k} = a_m e_k^{l_k-m}$, where a_m is a nonzero coefficient, it implies $0 = E^m E^{d-m} = a_m e_k^{l_k}$, which is a contradiction. In the same way, assume that $E^s\alpha_{l_k} = a_s e_{i+j+k}^{l_k+s}$; this means that $F^{s+1}E^s\alpha_{l_k} = 0$, and therefore, in view of remark 1) in step 5.2.2, we have $F^sE^s\alpha_{l_k} = b_s e_{i+j+k}^{l_k}$, where $b_s \in \mathbb{C}$. Applying the formula (see section 2) for the commutator $[E^s, F^s]$, we arrive to the conclusion that $\alpha_{l_k} = c_s e_{i+j+k}^{l_k}$, which is impossible.
2) Now the module over u_q generated by α_{l_k} is an extension of $P_{i+j+k} \oplus \mathrm{u}_q^+\alpha_{l_k}$,

hence they are both compelled to be extendable (see proposition 4.2). As in the proof of theorem 5.1, $\mathrm{u}_q^+\alpha_{l_k}$ must be isomorphic to $M^m_{i+j+k+l_k}$, with $m = d-1$ or $m \equiv -2(i+j+k+l_k) \mathrm{mod}\, d$. In order to exclude the second possibility, we suppose that $l_k = d - 2(i+j+k) + 1$; this means that $m \equiv -2(i+j+k+d-2(i+j+k)+1) \equiv -2(d-(i+j+k)+1) \equiv 2(i+j+k) - 2 \mathrm{mod}\, d$. In this case, the vectors $E^m\alpha_{l_k}$ and e^{d-1}_{i+j} are in the kernel of the morphism induced by the action of E on the vector spaces concerned. The fact that the kernel is one-dimensional gives a contradiction and therefore $\mathrm{u}_q^+\alpha_{l_k} = P_{i+j+k+l_k}$.

Step 5.2.4 *The* u_q*-module* $\overline{P_{i+j+k} \oplus P_{i+j+k+l_k}}$ *is indecomposable.*

Proof : Suppose it admits a non trivial decomposition $\overline{P_{i+j+k} \oplus P_{i+j+k+l_k}} = A \oplus B$, with A and B non zero. This implies that as u_q^+-modules (as such we denote them by $\underline{A}$ and $\underline{B}$) $\underline{A}$ or $\underline{B}$ is equal to P_{i+j+k}, and $\underline{B}$ or $\underline{A}$ is equal to P_{i+j+l_k} (by the Krull-Schmidt theorem). Hence A and B are extended u_q^+-projective modules, which is excluded. □

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